On the intrinsic geometry of a unit vector field*

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Abstract

We study the geometrical properties of a unit vector field on a Riemannian 2-manifold, considering the field as a local imbedding of the manifold into its tangent sphere bundle with the Sasaki metric. For the case of constant curvature K, we give a description of the totally geodesic unit vector fields for $K = 0$ and $K = 1$ and prove a non-existence result for $K \neq 0, 1$. We also found a family ξ_ω of vector fields on the hyperbolic 2-plane L^2 of curvature $-c^2$ which generate foliations on T_1L^2 with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$.

Keywords: Sasaki metric, vector field, sectional curvature, totally geodesic submanifolds.

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Introduction

A unit vector field ξ on a Riemannian manifold M is called *holonomic* if ξ is a field of normals of some family of regular hypersurfaces in M and *non-holonomic* otherwise. The geometry of non-holonomic unit vector fields has been developed by A.Voss at the end of the 19-th century. The foundations of this theory can be found in [1]. Recently, the geometry of a unit vector field has been considered from another point of view. Namely, let T_1M be the unit tangent sphere bundle of M endowed with the Sasaki metric [9]. If ξ is a unit vector field on M, then one may consider ξ as a mapping $\xi : M \to T_1M$ so that the image $\xi(M)$ is a submanifold in T_1M with the metric induced from T_1M. So, one may apply the methods from the study of the geometry of submanifolds to determine geometrical characteristics of a unit vector field. For example, the unit vector field ξ is said to be *minimal* if $\xi(M)$ is of minimal volume with respect to the induced metric [6]. A number of examples of locally minimal vector unit fields has been found (see [2, 3, 7]). On the other hand, using the geometry of submanifolds, we may find the Riemannian, Ricci or scalar curvature of a unit vector field using the second fundamental form of the submanifold $\xi(M) \in T_1M$ found in [11]. In this paper we apply this approach to the simplest case when the base space is 2-dimensional and hence the submanifold $\xi(M) \in T_1M$ is a hypersurface.

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1 The results

Let ξ be a given unit vector field. Denote by e_0 a unit vector field such that $\nabla_{e_0}\xi = 0$. Denote by e_1 a unit vector field, orthogonal to e_0, such that

$$\nabla_{e_1}\xi = \lambda\eta,$$

where η is a unit vector field, orthogonal to ξ. The function λ is a *signed* singular value of a linear operator $\nabla\xi : TM \to \xi^{\perp}$ (acting as $(\nabla\xi)X = \nabla_X\xi$). Set

$$\nabla_\xi\,\xi = k\,\eta, \quad \nabla_\eta\,\eta = \kappa\,\xi.$$

The functions k and κ are the *signed* geodesic curvatures of the integral curves of the fields ξ and η respectively. We prove that $\lambda^2 = k^2 + \kappa^2$.

Denote the *signed* geodesic curvatures of the integral curves of the fields e_0 and e_1 as μ and σ respectively. Then

$$\nabla_{e_0}\,e_0 = \mu\,e_1, \quad \nabla_{e_1}\,e_1 = \sigma\,e_0.$$

The rientations of the frames (ξ,η) and (e_0,e_1) are independent. Set $s = 1$ if the orientations are coherent and $s = 0$ otherwise.

The following result (Lemma 3.2) is a basic tool for the study.

Let M be a 2-dimensional Riemannian manifold of Gaussian curvature K. The second fundamental form Ω of the submanifold $\xi(M) \subset T_1M$ is given by

$$\Omega = \begin{bmatrix} -\mu\,\dfrac{\lambda}{\sqrt{1+\lambda^2}} & (-1)^{s+1}\dfrac{K}{2} + \dfrac{e_0(\lambda)}{1+\lambda^2} \\ (-1)^{s+1}\dfrac{K}{2} + \dfrac{e_0(\lambda)}{1+\lambda^2} & e_1\left(\dfrac{\lambda}{\sqrt{1+\lambda^2}}\right) \end{bmatrix}.$$

Using the formula for the sectional curvature of T_1M^n, we find an expression for the Gaussian curvature of $\xi(M^2)$ (Lemma 3.4).

The Gaussian curvature K_ξ of a hypersurface $\xi(M) \in T_1M$ is given by

$$K_\xi = \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} + (-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K) + \frac{1}{2}\mu e_1\left(\frac{1}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2,$$

where K is the Gaussian curvature of M.

As applications of these Lemmas, we prove the following theorems.

Theorem 3.1 *Let M^2 be a Riemannian manifold of constant Gaussian curvature K. A unit vector field ξ generating a totally geodesic submanifold in T_1M^2 exists if and only if $K = 0$ or $K = 1$. Moreover,* *(a) if $K = 0$, then ξ is either a parallel vector field or moving along a family of parallel geodesics with constant angle speed. Geometrically, $\xi(M^2)$ is either M^2 imbedded isometrically into $M^2 \times S^1$ as a factor or a (helical) flat submanifold in $M^2 \times S^1$;*

(b) if $K = 1$, then ξ is a vector field on a standard sphere S^2 which is parallel along the meridians and moving along the parallels with a unit angle speed. Geometrically, $\xi(M^2)$ is a part of totally geodesic RP^2 locally isometric to sphere S^2 of radius 2 in $T_1S^2 \overset{isom}{\approx} RP^3$.

Theorem 3.3 *Let M^2 be a space of constant Gaussian curvature K. Suppose that ξ is a unit geodesic vector field on M^2. Then $\xi(M^2)$ has constant Gaussian curvature in one of the following cases:*

(a) $K = -c^2 < 0$ and ξ is a normal vector field for the family of horocycles on the hyperbolic 2-plane L^2 of curvature $-c^2$. In this case, $K_\xi = -c^2$ and therefore $\xi(M^2)$ is locally isometric the base space;

(b) $K = 0$ and ξ is a parallel vector field on M^2. In this case $K_\xi = 0$ and $\xi(M^2)$ is also locally isometric to the base space;

(c) $K = 1$ and ξ is any (local) geodesic vector field on the standard sphere S^2. In this case, $K_\xi = 0$.

Theorem 3.4 *Let L^2 be a hyperbolic 2-plane of constant curvature $-c^2$. Then T_1L^2 admits a hyperfoliation with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$. The leaves are generated by unit vector fields making a constant angle with a pencil of parallel geodesics on L^2.*

2 Basic definitions and preliminary results

Let (M, g) be an $(n+1)$ – dimensional Riemannian manifold with metric g. Let ∇ denote the Levi-Civita connection on M. Then $\nabla_X \xi$ is always orthogonal to ξ and hence, $(\nabla\xi)X \overset{def}{=} \nabla_X \xi : T_pM \to \xi_p^\perp$ is a linear operator at each $p \in M$. We define an adjoint operator $(\nabla\xi)^* X : \xi_p^\perp \to T_pM$ by

$$\langle (\nabla\xi)^* X, Y \rangle_g = \langle X, \nabla_Y \xi \rangle_g .$$

Then there is an orthonormal frame $e_0, e_1, \dots, e_n$ in T_pM and an orthonormal frame $f_1, \dots, f_n$ in $\xi_p^\perp$ such that

$$(\nabla\xi)e_0 = 0, \quad (\nabla\xi)e_\alpha = \lambda_\alpha f_\alpha, \quad (\nabla\xi)^* f_\alpha = \lambda_\alpha e_\alpha, \qquad \alpha = 1, \dots, n, \tag{1}$$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are real-valued functions.

Definition 2.1 *The orthonormal frames satisfying (1) are called* singular frames *for the linear operator $(\nabla\xi)$ and the real valued functions $\lambda_1, \lambda_2, \dots \lambda_n$ are called the (signed) singular values of the operator $\nabla\xi$ with respect to the singular frame.*

Remark that the sign of the singular value is definied up to the directions of the vectors of the singular frame.

For each $\tilde{X} \in T_{(p,\xi)}TM$ there is a decomposition

$$\tilde{X} = X_1^h + X_2^v$$

where $(\cdot)^h$ and $(\cdot)^v$ are the horizontal and vertical lifts of vectors X_1 and X_2 from T_pM to $T_{(p,\xi)}TM$. The Sasaki metric is defined by the scalar product of the form

$$\big\langle\big\langle \tilde{X}, \tilde{Y} \big\rangle\big\rangle = \big\langle X_1, Y_1 \big\rangle + \big\langle X_2, Y_2 \big\rangle,$$

where $\big\langle \cdot, \cdot \big\rangle$ means the scalar product with respect to metric g.

The following lemma has been proved in [11].

Lemma 2.1 *At each point $(p, \xi) \in \xi(M) \subset TM$ the vectors*

$$\begin{cases} \tilde{e}_0 = e_0^h, \\ \tilde{e}_\alpha = \dfrac{1}{\sqrt{1+\lambda_\alpha^2}}(e_\alpha^h + \lambda_\alpha f_\alpha^v), \qquad \alpha = 1, \dots, n, \end{cases} \tag{2}$$

form an orthonormal frame in the tangent space of $\xi(M)$ and the vectors

$$\tilde{n}_{\sigma|} = \frac{1}{\sqrt{1+\lambda_\sigma^2}}\big(-\lambda_\sigma e_\sigma^h + f_\sigma^v \,\big), \qquad \sigma = 1, \dots, n, \tag{3}$$

form an orthonormal frame in the normal space of $\xi(M)$.

Let $R(X,Y)\xi = [\nabla_X, \nabla_Y]\,\xi - \nabla_{[X,Y]}\,\xi$ be the curvature tensor of M. Introduce the following notation

$$r(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi. \tag{4}$$

Then, evidently,

$$R(X,Y)\xi = r(X,Y)\xi - r(Y,X)\xi.$$

The following Lemma has also been proved in [11].

Lemma 2.2 *The components of second fundamental form of $\xi(M) \subset T_1M$ with respect to the frame (3) are given by*

$$\begin{aligned}
\tilde{\Omega}_{\sigma|00} &= \tfrac{1}{\sqrt{1+\lambda_\sigma^2}}\big\langle r(e_0, e_0)\xi, f_\sigma \big\rangle, \\
\tilde{\Omega}_{\sigma|\alpha 0} &= \tfrac{1}{2}\tfrac{1}{\sqrt{(1+\lambda_\sigma^2)(1+\lambda_\alpha^2)}}\Big[\big\langle r(e_\alpha, e_0)\xi + r(e_0, e_\alpha)\xi, f_\sigma \big\rangle + \\
&\qquad \lambda_\sigma \lambda_\alpha \big\langle R(e_\sigma, e_0)\xi, f_\alpha \big\rangle\Big], \\
\tilde{\Omega}_{\sigma|\alpha\beta} &= \tfrac{1}{2}\tfrac{1}{\sqrt{(1+\lambda_\sigma^2)(1+\lambda_\alpha^2)(1+\lambda_\beta^2)}}\Big[\big\langle r(e_\alpha, e_\beta)\xi + r(e_\beta, e_\alpha)\xi, f_\sigma \big\rangle \\
&\qquad + \lambda_\alpha \lambda_\sigma \big\langle R(e_\sigma, e_\beta)\xi, f_\alpha \big\rangle + \lambda_\beta \lambda_\sigma \big\langle R(e_\sigma, e_\alpha)\xi, f_\beta \big\rangle\Big],
\end{aligned}$$

where $\{e_0, e_1, \dots, e_n; f_1, \dots, f_n\}$ is a singular frame of $(\nabla\xi)$ and $\lambda_1, \dots, \lambda_n$ are the corresponding singular values.

Let $\tilde\nabla$ and ∇ be the Levi-Civita connections of the Sasaki metric of TM and the metric of M respectively. The Kowalski formulas [8] give the covariant derivatives of combinations of lifts of vector fields.

Lemma 2.3 (O.Kowalski) *Let X and Y be vector fields on M. Then at each point $(p,\xi)\in TM$ we have*

$$\begin{aligned}
&\tilde\nabla_{X^h}Y^h = (\nabla_X Y)^h - \tfrac{1}{2}\left(R(X,Y)\xi\right)^v,\\
&\tilde\nabla_{X^h}Y^v = \tfrac{1}{2}\left(R(\xi,Y)X\right)^h + (\nabla_X Y)^v,\\
&\tilde\nabla_{X^v}Y^h = \tfrac{1}{2}\left(R(\xi,X)Y\right)^h,\\
&\tilde\nabla_{X^v}Y^v = 0,
\end{aligned}$$

where R is the Riemannian curvature tensor of (M,g).

This basic result allows to find the curvature tensor of TM (see [8]) and the curvature tensor of T_1M (see [4]). As a corollary, it is not too hard to find an expression for the *sectional curvature* of T_1M. It is well-known that ξ^v is a unit normal for T_1M as a hypersurface in TM. Thus, $\tilde X = X_1^h + X_2^v$ is tangent to T_1M if and only if $\langle X_2,\xi\rangle = 0$.

Let $\tilde X = X_1^h + X_2^v$ and $\tilde Y = Y_1^h + Y_2^v$, where $X_2, Y_2\in \xi^\perp$, form an orthonormal base of a 2-plane $\tilde\pi\subset T_{(p,\xi)}T_1M$. Then we have [5]:

$$\begin{aligned}
\tilde K(\tilde\pi) = \ & \big\langle R(X_1,Y_1)Y_1,X_1\big\rangle - \tfrac{3}{4}\|R(X_1,Y_1)\xi\|^2 + \\
& \tfrac{1}{4}\|R(\xi,Y_2)X_1 + R(\xi,X_2)Y_1\|^2 + \|X_2\|^2\|Y_2\|^2 - \big\langle X_2,Y_2\big\rangle^2 + \\
& 3\big\langle R(X_1,Y_1)Y_2,X_2\big\rangle - \big\langle R(\xi,X_2)X_1, R(\xi,Y_2)Y_1\big\rangle + \\
& \big\langle(\nabla_{X_1}R)(\xi,Y_2)Y_1,X_1\big\rangle + \big\langle(\nabla_{Y_1}R)(\xi,X_2)X_1,Y_1\big\rangle.
\end{aligned} \tag{5}$$

Combining the results of Lemma 2.1, Lemma 2.2 and (5), we can write an expression for the sectional curvature of $\xi(M)$.

Lemma 2.4 *Let $\tilde X$ and $\tilde Y$ be an ortonormal vectors which span a 2-plane $\tilde\pi$ tangent to $\xi(M)\subset T_1M$. Denote by $K_\xi(\tilde\pi)$ the sectional curvature $\xi(M)$ with respect to metric, induced by Sasaki metric of T_1M. Then*

$$K_\xi(\tilde\pi) = \tilde K(\tilde\pi) + \sum_\sigma\Big(\Omega_{\sigma|}(\tilde X,\tilde X)\Omega_{\sigma|}(\tilde Y,\tilde Y) - \Omega^2_{\sigma|}(\tilde X,\tilde Y)\Big), \tag{6}$$

where $\tilde K(\tilde\pi)$ is the sectional curvature of T_1M given by (5), $\Omega_{|\sigma}$ are the components of the second fundamental form of $\xi(M)$ given by Lemma 2.2 and the vectors are given with respect to the frame (2).

3 The 2-dimensional case

Let M be a 2-dimensional Riemannian manifold. The following proposition gives useful information about the relation between the singular values of the $(\nabla\xi)$- operator, geometric characteristics of the integral curves of singular frame and the Gaussian curvature of the manifold.

Lemma 3.1 *Let ξ be a given smooth unit vector field on M^2. Denote by e_0 a unit vector field on M^2 such that $\nabla_{e_0}\xi = 0$. Let η and e_1 be the unit vector fields on M^2 such that (ξ,η) and (e_0,e_1) form two orthonormal frames on M^2. Denote by λ a signed singular value of the operator $(\nabla\xi)$. Then we have*

$$\nabla_{e_1}\xi = \lambda\eta,$$

and the following relations hold:

(a) if $k = \langle\nabla_\xi\xi,\eta\rangle$ is a signed geodesic curvature of a ξ-curve and $\kappa = \langle\nabla_\eta\eta,\xi\rangle$ is a signed geodesic curvature of a η-curve, then

$$\lambda^2 = k^2 + \kappa^2;$$

(b) if K is the Gaussian curvature of M^2, then

$$(-1)^s K = e_0(\lambda) - \lambda\sigma,$$

where $\sigma = \langle\nabla_{e_1}e_1, e_0\rangle$ is a signed geodesic curvature of a e_1-curve and

$$s = \begin{cases} 1 \text{ if the frames } (\xi,\eta) \text{ and } (e_0,e_1) \text{ have the same orientation,} \\ 0 \text{ if the frames } (\xi,\eta) \text{ and } (e_0,e_1) \text{ have an opposite orientation} \end{cases}.$$

Proof. (a) If (ξ,η) is an orthonormal frame on M^2, then

$$\begin{aligned} &\nabla_\xi\,\xi = k\,\eta, &&\nabla_\xi\,\eta = -k\,\xi, \\ &\nabla_\eta\,\xi = -\kappa\,\eta, &&\nabla_\eta\,\eta = \kappa\,\xi. \end{aligned} \tag{7}$$

Geometrically, the functions k and κ are the signed geodesic curvatures of ξ- and η-curves respectively.

In a similar way we get

$$\begin{aligned} &\nabla_{e_0}e_0 = \mu\,e_1, &&\nabla_{e_0}e_1 = -\mu\,e_0, \\ &\nabla_{e_1}e_0 = -\sigma\,e_1, &&\nabla_{e_1}e_1 = \sigma\,e_0, \end{aligned} \tag{8}$$

where μ and σ are the signed geodesic curvatures of the e_0- and e_1-curves respectively.

Let ω be an angle function between ξ and e_0. Then we have two possible decompositions:

$$\mathrm{Or}(+)\begin{cases} e_0 = \cos\omega\,\xi + \sin\omega\,\eta, \\ e_1 = -\sin\omega\,\xi + \cos\omega\,\eta, \end{cases} \qquad \mathrm{Or}(-)\begin{cases} e_0 = \cos\omega\,\xi + \sin\omega\,\eta, \\ e_1 = \sin\omega\,\xi - \cos\omega\,\eta. \end{cases}$$

In the case $Or(+)$ we have

$$\nabla_{e_0}\xi = (k\cos\omega - \kappa\sin\omega)\,\eta,$$
$$\nabla_{e_1}\xi = -(k\sin\omega + \kappa\cos\omega)\,\eta,$$

and due to the choice of e_0 and e_1 we see that

$$\left\{\begin{array}{lcl} k\cos\omega - \kappa\sin\omega & = & 0, \\ k\sin\omega + \kappa\cos\omega & = & -\lambda. \end{array}\right.$$

So, for the case of $Or(+)$ $\;k = -\lambda\sin\omega,\; \kappa = -\lambda\cos\omega$.

In a similar way, for the case of $Or(-)$ $\;k = \lambda\sin\omega,\; \kappa = \lambda\cos\omega$. In both cases

$$\lambda^2 = k^2 + \kappa^2.$$

(b) Due to the choice of the frames,

$$\begin{aligned}\big\langle R(e_0,e_1)\xi,\eta\big\rangle = \;& \big\langle \nabla_{e_0}\nabla_{e_1}\xi - \nabla_{e_1}\nabla_{e_0}\xi - \nabla_{\nabla_{e_0}e_1-\nabla_{e_1}e_0}\xi,\eta\big\rangle = \\ & \big\langle \nabla_{e_0}(\lambda\,\eta) - \nabla_{-\mu\,e_0+\sigma\,e_1}\,\xi,\eta\big\rangle = e_0(\lambda) - \lambda\sigma.\end{aligned}$$

On the other hand,

$$\big\langle R(e_0,e_1)\xi,\eta\big\rangle = \left\{\begin{array}{l} -K \text{ for the case of } Or(+), \\ +K \text{ for the case of } Or(-). \end{array}\right. \tag{9}$$

Set $s = 1$ for the case $Or(+)$ and $s = 0$ for the case $Or(-)$. Combining the results, we get $(-1)^s K = e_0(\lambda) - \lambda\sigma$, which completes the proof.

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The result of Lemma 2.2 can also be simplified in the following way.

Lemma 3.2 *Let M be a 2-dimensional Riemannian manifold of Gaussian curvature K. In terms of Lemma 3.1 the second fundamental form of the submanifold $\xi(M)\subset T_1M$ can be presented in two equivalent forms:*

(i)

$$\Omega = \begin{bmatrix} -\mu\,\dfrac{\lambda}{\sqrt{1+\lambda^2}} & (-1)^{s+1}\dfrac{K}{2} + \dfrac{e_0(\lambda)}{1+\lambda^2} \\ (-1)^{s+1}\dfrac{K}{2} + \dfrac{e_0(\lambda)}{1+\lambda^2} & e_1\left(\dfrac{\lambda}{\sqrt{1+\lambda^2}}\right) \end{bmatrix},$$

(ii)

$$\Omega = \begin{bmatrix} -\mu\,\dfrac{\lambda}{\sqrt{1+\lambda^2}} & \dfrac{1}{2}\left(\sigma\,\lambda + \dfrac{1-\lambda^2}{1+\lambda^2}e_0(\lambda)\right) \\ \dfrac{1}{2}\left(\sigma\,\lambda + \dfrac{1-\lambda^2}{1+\lambda^2}e_0(\lambda)\right) & e_1\left(\dfrac{\lambda}{\sqrt{1+\lambda^2}}\right) \end{bmatrix}.$$

Proof.
At each point $(p, \xi) \in \xi(M)$ the vectors

$$\begin{cases} \tilde{e}_0 = e_0^h, \\ \tilde{e}_1 = \dfrac{1}{\sqrt{1+\lambda^2}}(e_1^h + \lambda \eta^v) \end{cases}$$

form an orthonormal frame in the tangent space of $\xi(M)$ and

$$\tilde{n} = \frac{1}{\sqrt{1+\lambda^2}}\big(-\lambda e_1^h + \eta^v \big),$$

is a unit normal for $\xi(M) \subset T_1M$.

Thus we see that in a 2-dimensional case the components of Ω take the form

$$\begin{aligned} \Omega_{00} &= \frac{1}{\sqrt{1+\lambda^2}}\big\langle r(e_0, e_0)\xi, \eta \big\rangle, \quad \Omega_{11} = \frac{1}{(1+\lambda^2)^{3/2}}\big\langle r(e_1, e_1)\xi, \eta \big\rangle, \\ \Omega_{01} &= \frac{1}{2}\frac{1}{1+\lambda^2}\Big[\big\langle r(e_1, e_0)\xi + r(e_0, e_1)\xi, \eta \big\rangle + \lambda^2 \big\langle R(e_1, e_0)\xi, \eta \big\rangle\Big]. \end{aligned}$$

Keeping in mind (4), (8) and (9), we see that

$$\begin{aligned} &\big\langle r(e_0, e_0)\xi, \eta \big\rangle = -\mu\lambda, &&\big\langle r(e_0, e_1)\xi, \eta \big\rangle = e_0(\lambda), \\ &\big\langle r(e_1, e_0)\xi, \eta \big\rangle = \sigma\lambda, &&\big\langle r(e_1, e_1)\xi, \eta \big\rangle = e_1(\lambda), \\ &\big\langle R(e_0, e_1)\xi, \eta \big\rangle = (-1)^s K. \end{aligned}$$

So we have

$$\Omega_{00} = -\mu \frac{\lambda}{\sqrt{1+\lambda^2}}, \qquad \Omega_{11} = \frac{e_1(\lambda)}{(1+\lambda^2)^{3/2}} = e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right),$$

$$\Omega_{01} = \frac{1}{2(1+\lambda^2)}(e_0(\lambda) + \lambda\sigma - \lambda^2(-1)^s K) = \begin{cases} (-1)^{s+1}\dfrac{K}{2} + \dfrac{e_0(\lambda)}{1+\lambda^2} \\ \dfrac{1}{2}\left(\sigma\,\lambda + \dfrac{1-\lambda^2}{1+\lambda^2}e_0(\lambda)\right) \end{cases},$$

where Lemma 3.1 (b) has been applied in two ways.

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3.1 Totally geodesic vector fields

The main goal of this section is to prove the following theorem.

Theorem 3.1 *Let M^2 be a Riemannian manifold of constant Gaussian curvature K. A unit vector field ξ generating a totally geodesic submanifold in T_1M^2 exists if and only if $K = 0$ or $K = 1$. Moreover,* *a) if $K = 0$, then ξ is either a parallel vector field or is moving along a family of parallel geodesics with constant angle speed. Geometrically, $\xi(M^2)$ is either M^2 imbedded isometrically into $M^2 \times S^1$ as a factor or a (helical) flat submanifold in $M^2 \times S^1$;*

b) if $K = 1$, then ξ is a vector field on a sphere S^2 which is parallel along the meridians and moving along the parallels with a unit angle speed. Geometrically, $\xi(M^2)$ is a part of totally geodesic RP^2 locally isometric to sphere S^2 of radius 2 in $T_1 S^2 \overset{isom}{\approx} RP^3$.

The proof will be divided into a series of separate propositions.

Proposition 3.1 *Let M^2 be a Riemannian manifold. Let D be a domain in M^2 endowed with a semi-geodesic coordinate system such that $ds^2 = du^2 + f^2\,dv^2$, where $f(u, v)$ is some non-vanishing function. Denote by (e_0, e_1) an orthonormal frame in D and specify $e_0 = \partial_u$, $e_1 = f^{-1}\partial_v$. If ξ is a unit vector field in D parallel along u-geodesics, then ξ can be written given as*

$$\xi = \cos\omega\, e_0 + \sin\omega\, e_1,$$

where $\omega = \omega(v)$ is an angle function and

(a) a singular frame for ξ may be chosen as $\{e_0,\ e_1, \eta = -\sin\omega\, e_0 + \cos\omega\, e_1\}$;

(b) a singular value for ξ in this case is $\lambda = e_1(\omega) - \sigma$, where σ is a signed geodesic curvature of the e_1-curves.

Proof. Indeed, if ξ is parallel along u-geodesics, then evidently the angle function ω between ξ and the u-curves does not depend on u. So this function has the form $\omega = \omega(v)$ and $\xi = \cos\omega\, e_0 + \sin\omega\, e_1$. Moreover, since

$$\begin{aligned} \nabla_{e_0} e_0 &= 0, & \nabla_{e_0} e_1 &= 0, \\ \nabla_{e_1} e_0 &= \tfrac{f_u}{f} e_1, & \nabla_{e_1} e_1 &= -\tfrac{f_u}{f} e_0, \end{aligned}$$

we see that $\sigma = -\frac{f_u}{f}$ and $\nabla_{e_1}\xi = (e_1(\omega) - \sigma)\,\eta$, where $\eta = -\sin\omega\, e_0 + \cos\omega\, e_1$. Therefore, $\lambda = e_1(\omega) - \sigma$ and the proof is complete.

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Proposition 3.2 *Let M^2 be a Riemannian manifold of constant negative curvature $K = -r^{-2} < 0$. Then there is no totally geodesic unit vector field on M^2.*

Proof. Suppose ξ is totally geodesic unit vector field on M^2. Set $\Omega \equiv 0$ in Lemma 3.2. Then $\lambda\mu \equiv 0$. If $\lambda \equiv 0$ in some domain $D \subset M^2$, then ξ is parallel in this domain and hence M^2 is flat in D, which contradicts the hypothesis. Suppose that $\mu \equiv 0$ at least in some domain $D \subset M^2$. This means that e_0-curves are geodesics in D and the field ξ is parallel along them. Choose a family of e_0-curves and the orthogonal trajectories as a local coordinate net in D. Then the first fundamental form of M^2 takes the form

$$ds^2 = du^2 + f^2\,dv^2,$$

where $f(u, v)$ is some function. Since M^2 is of constant curvature $K = -\dfrac{1}{r^2}$, the function f satisfies the equation

$$f_{uu} - \frac{1}{r^2} f = 0.$$

The general solution of this equation is

$$f(u, v) = A(v) \cosh(u/r) + B(v) \sinh(u/r).$$

There are two possible cases:

$$\begin{array}{rl} (i) & A^2(v) \equiv B^2(v) \text{ over the whole domain } D; \\ (ii) & A^2(v) \neq B^2(v) \text{ in some subdomain } D' \subset D. \end{array}$$

Case (i). In this case, in dependence of the signs of $A(v)$ and $B(v)$,

$$f(u, v) = A(v)e^{u/r} \quad \text{or} \quad f(u, v) = A(v)e^{-u/r}.$$

Consider the first case (the second case can be reduced to the first one after the parameter change $u \to -u$). Making an evident v-parameter change, we reduce the metric to the form

$$ds^2 = du^2 + r^2 e^{2\,u/r}\, dv^2.$$

Applying Proposition 3.1 for $f = re^{u/r}$, we get $\lambda = \dfrac{1}{r}(\omega'\, e^{-u/r} + 1)$. Setting $\Omega_{11} \equiv 0$, we see that $e_1(\lambda) \equiv 0$. Hence $\omega'' = 0$, i.e., $\omega = av + b$. Therefore,

$$\lambda = \frac{1}{r}\left(a\, e^{-u/r} + 1\right).$$

Considering $\Omega_{01} \equiv 0$ (with $s = 1$ because of $Or(+)$-case), we get

$$-\frac{1}{2r^2} + \frac{\frac{1}{r} e_0(a\, e^{-u/r} + 1)}{1 + \frac{1}{r^2}(e^{-u/r}a + 1)^2} = -\frac{(\frac{1}{r^2} + 1)(ae^{-u/r} + 1)^2 - a^2 e^{-2u/r}}{2\, r^2[1 + \frac{1}{r^2}(ae^{-u/r} + 1)^2]} \not\equiv 0,$$

and hence, this case is not possible.

Case (ii). Choose a subdomain $D' \subset D$ such that $A^2(v) < B^2(v)$ or $A^2(v) > B^2(v)$ over D'. Then the function f may be presented respectively in two forms:

$$\begin{array}{rl} \text{(a)} & f(u, v) = \sqrt{B^2 - A^2}\, \sinh(u/r + \theta) \quad \text{or} \\ \text{(b)} & f(u, v) = \sqrt{A^2 - B^2}\, \cosh(u/r + \theta), \end{array}$$

where $\theta(v)$ is some function.

Consider the case (a). After a v-parameter change, the metric in D' takes the form

$$ds^2 = du^2 + r^2 \sinh^2(u/r + \theta)\, dv^2.$$

Applying Proposition 3.1 for $f = r\sinh(u/r+\theta)$, we get

$$\lambda = \frac{\omega'}{r\,\sinh(u/r+\theta)} + \frac{1}{r}\coth(u/r+\theta).$$

Considering $\Omega_{11} \equiv 0$, we have $e_1(\lambda) \equiv 0$ which implies the identity

$$\omega''\,\sinh(u/r+\theta) - \omega'\theta'\,\cosh(u/r+\theta) - \theta' \equiv 0.$$

From this we get $\omega'' = 0, \quad \theta' = 0$ and hence $\left\{\begin{array}{l}\theta = const,\\ \omega = av+b\end{array}\right.$ $(a, b = const)$. After a parameter change we reduce the metric to the form

$$ds^2 = du^2 + r^2\sinh^2(u/r)\,dv^2$$

Applying Proposition 3.1 for $f = r\sinh(u/r)$, we get $\lambda = \dfrac{a+\cosh(u/r)}{r\sinh(u/r)}$. The substitution into Ω_{01} gives

$$-\frac{1}{2}\frac{(\frac{1}{r^2}+1)[a+\cosh(u/r)]^2 - a^2 + 1}{r^2\sinh^2(u/r) + [a+\cosh(u/r)]^2} \not\equiv 0,$$

which completes the proof for the polar case.

The *Cartesian* case consideration gives $\omega = av+b, \quad \lambda = \dfrac{a+\sinh(u/r)}{r\cosh(u/r)}$ and $\Omega_{01} = -\dfrac{1}{2}\dfrac{(\frac{1}{r^2}+1)[a+\sinh(u/r)]^2 - a^2 - 1}{r^2\cosh^2(u/r) + [a+\sinh(u/r)]^2} \not\equiv 0$, which completes the proof. ■

Proposition 3.3 *Let M^2 be a Riemannian manifold of constant positive curvature $K = r^{-2} > 0$. Then a totally geodesic unit vector field ξ on M^2 exists if $r = 1$ and ξ is parallel along the meridians of M^2 locally isometric to S^2 and moves along the parallels with a unit angle speed. Geometrically, $\xi(M^2)$ is a part of totally geodesic RP^2 locally isometric to sphere S^2 of radius 2 in $T_1S^2 \overset{isom}{\approx} RP^3$.*

Proof. Suppose ξ is totally geodesic unit vector field on M^2. The same arguments as in Proposition 3.2 lead to the case $\mu \equiv 0$ at least in some domain $D \subset M^2$. So, choose again a family of e_0-curves and the orthogonal trajectories as a local coordinate net in D. Then the first fundamental form of M^2 can be expressed as $ds^2 = du^2 + f^2\,dv^2$, where $f(u, v)$ is some function. Since M^2 is of constant curvature $K = r^{-2}$, the function f satisfies the equation

$$f_{uu} + \frac{1}{r^2}f = 0.$$

The general solution of this equation $f(u, v) = A(v)\cos(u/r) + B(v)\sin(u/r)$ may be presented in two forms:

$$\begin{array}{ll}\text{(a)} & f(u,v) = \sqrt{A^2+B^2}\,\sin(u/r+\theta) \quad \text{or}\\ \text{(b)} & f(u,v) = \sqrt{A^2+B^2}\,\cos(u/r+\theta),\end{array}$$

where $\theta(v)$ is some function.

Consider first, the case (a). After v-parameter change, the metric in D takes the form

$$ds^2 = du^2 + r^2 \sin^2(u/r + \theta)\, dv^2.$$

Applying Proposition 3.1 for $f = r\sin(u/r + \theta)$, we get

$$\lambda = \frac{\omega'}{r\,\sin(u/r + \theta)} + \frac{1}{r}\cot(u/r + \theta).$$

Setting $\Omega_{11} \equiv 0$, we find $e_1(\lambda) \equiv 0$ which implies the identity

$$\omega''\,\sin(u/r + \theta) - \omega'\theta'\,\cos(u/r + \theta) + \theta' \equiv 0.$$

From this $\omega'' = 0$, $\theta' = 0$ and we have again $\begin{cases} \theta = const, \\ \omega = av + b \end{cases}$ $a, b = const.$ After a suitable u-parameter change, we reduce the metric to the form

$$ds^2 = du^2 + r^2 \sin^2(u/r)\, dv^2$$

Applying Proposition 3.1 for $f = r\sin(u/r)$, we get $\lambda = \dfrac{a + \cos(u/r)}{r\,\sin(u/r)}$. Substitution into Ω_{01} gives

$$\frac{1}{2}\frac{(\frac{1}{r^2} - 1)[a + \cos(u/r)]^2 + a^2 - 1}{r^2\sin^2(u/r) + [a + \cos(u/r)]^2} \equiv 0,$$

which is possible only if $r = 1$ and $|a| = 1$. So, we obtain to the standard sphere metric

$$ds^2 = du^2 + \sin^2 u\, dv^2$$

and (after the $\pm v + b \to v$ parameter change) the unit vector field

$$\xi = \left\{\cos v, \frac{\sin v}{\sin u}\right\}.$$

This vector field is parallel along the meridians of S^2 and moves helically along the parallels of S^2 with unit angle speed.

For the case (b) one can find $\omega = av + b$, $\quad \lambda = \dfrac{a - \sin(u/r)}{r\,\cos(u/r)}$ and

$$\Omega_{01} = \frac{1}{2}\frac{(\frac{1}{r^2} - 1)[a - \sin(u/r)]^2 + a^2 - 1}{r^2\cos^2(u/r) + [a - \sin(u/r)]^2} \equiv 0,$$

which gives $r = 1$ and $|a| = 1$ as a result. Thus, we have a metric

$$ds^2 = du^2 + \cos^2 u\, dv^2$$

and a vector field $\xi = \left\{\cos v, \dfrac{\sin v}{\cos u}\right\}$. It is easy to see that the results of cases (a) and (b) are geometrically equivalent.

Introduce the local coordinates (u, v, ω) on T_1S^2, where ω is the angle between arbitrary unit vector ξ and the coordinate vector field $X_1 = \{1, 0\}$. The first fundamental form of T_1S^2 with respect to these coordinates is [10]

$$d\tilde{s}^2 = du^2 + dv^2 + 2\cos u\, dv\, d\omega + d\omega^2.$$

The local parameterization of the submanifold $\xi(S^2)$, generated by the given field, is $\omega = v$ and the induced metric on $\xi(S^2)$ is

$$d\tilde{s}^2 = du^2 + 2(1 + \cos u)\, dv^2 = du^2 + 4\cos^2 u/2\, dv^2.$$

Thus, $\xi(S^2)$ is locally isometric to sphere S^2 of radius 2. Since $T_1S^2 \overset{isom}{\approx} RP^3$ and there are no other totally geodesic submanifolds in RP^3 except RP^2, we see that $\xi(S^2)$ is a part of RP^2 . So the proof is complete.

■

Proposition 3.4 *Let M^2 be a Riemannian manifold of constant zero curvature $K = 0$. Then a totally geodesic unit vector field ξ on M^2 is either parallel or moves along the family of parallel geodesics with constant angle speed. Geometrically, $\xi(M^2)$ is either E^2 imbedded isometrically into $E^2 \times S^1$ as a factor or a helical flat submanifold in $E^2 \times S^1$.*

Proof. Suppose ξ is totally geodesic unit vector field on M^2. Set $\Omega \equiv 0$ in Lemma 3.2. Then $\lambda\mu \equiv 0$. If $\lambda \equiv 0$ over some domain $D \subset M^2$, then ξ is *parallel in this domain.*

Suppose $\lambda \not\equiv 0$ in a domain $D \subset M^2$. Then $\mu \equiv 0$ on at least a subdomain $D' \subset D$. This means that the e_0-curves are geodesics in D' and the field ξ is parallel along them. Choose a family of e_0-curves and the orthogonal trajectories as a local coordinate net in D'. Then the first fundamental form of M^2 takes the form $ds^2 = du^2 + f^2\, dv^2$ and since M^2 is of zero curvature, f satisfies the equation

$$f_{uu} = 0.$$

A general solution of this equation is $f(u, v) = A(v)u + B(v)$. There are two possible cases:

(a) $A(v) \neq 0$ in some subdomain $D'' \subset D'$;
(b) $A(v) \equiv 0$ over the whole domain D'.

Case(a). The function f may be presented over D'' in the form

$$f(u, v) = A(v)(u + \theta),$$

where $\theta(v) = B(v)/A(v)$. After a v-parameter change, the metric in D'' takes the form $ds^2 = du^2 + (u + \theta)^2\, dv^2$. Applying Proposition 3.1 for $f = u + \theta$, we get $\lambda = \dfrac{\omega' + 1}{u + \theta}$. Setting $\Omega_{11} \equiv 0$, we obtain the identity

$$\omega''(u + \theta) - (\omega' + 1)\theta' \equiv 0.$$

From this we get $\left\{\begin{array}{l}\omega''=0\\ \omega'=-1\end{array}\right.$ or $\left\{\begin{array}{l}\omega''=0\\ \theta'=0\end{array}\right.$. In the first case, $\lambda=0$ and the field ξ is parallel again. In the second case $\left\{\begin{array}{l}\theta=const,\\ \omega=av+b\end{array}\right.$ $a,b=const.$

Making a parameter change, we reduce the metric to the form

$$ds^2=du^2+u^2\,dv^2$$

Applying Proposition 3.1 with $f(u,v)=u$, we get $\lambda=\dfrac{a+1}{u}$. The substitution into Ω_{01} gives the condition

$$-\frac{a+1}{u^2+(a+1)^2}=0$$

which is possible only if $a=-1$. But this means that again $\lambda=0$ and hence ξ is a parallel vector field.

Case (b). After a v-parameter change, the metric takes the form

$$ds^2=du^2+dv^2.$$

Applying Proposition 3.1 for $f\equiv 1$, we get $\lambda=\omega'$. Setting $\Omega_{11}\equiv 0$, we find $\omega''\equiv 0$. This means that $\omega=av+b$ and ξ is either parallel along the u-lines $(a=0)$ or moves along the u-lines helically with constant angle speed.

Let (u,v,ω) be standard coordinates in $E^2\times S^1$. Then the first fundamental form of $E^2\times S^1$ is

$$d\tilde{s}^2=du^2+dv^2+d\omega^2.$$

If $a=0$, then with respect to these coordinates the local parameterization of $\xi(E^2)$ is $\omega=const$ and $\xi(E^2)$ is nothing else but E^2 isometrically imbedded into $E^2\times S^1$. If $a\neq 0$, then the local parameterization of $\xi(E^2)$ is $\omega=av+b$ and the induced metric is

$$d\tilde{s}^2=du^2+(1+a^2)\,dv^2$$

which is flat. The imbedding is helical in the sense that this submanifold meets each flat element of the cylinder $p: E^2\times S^1\to S^1$ under constant angle $\varphi=\arccos\frac{1}{\sqrt{1+a^2}}$. So the proof is complete.

■

3.2 The curvature

The main goal of this section is to obtain an explicit formula for the Gaussian curvature of $\xi(M^2)$ and apply it to some specific cases. The first step is the following lemma.

Lemma 3.3 *Let ξ be a unit vector field on a 2-dimensional Riemannian manifold of Gaussian curvature K. In terms of Lemma 3.1, the sectional curvature $K_{T_1M}(\xi)$ of T_1M along 2-planes tangent to $\xi(M)$ is given by*

$$K_{T_1M}(\xi)=\frac{K^2}{4}+\frac{K(1-K)}{1+\lambda^2}+(-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K).$$

Proof. Let $\tilde{\pi}$ be a 2-plane tangent to $\xi(M)$. Then $\tilde{X} = e_0^h$ and $\tilde{Y} = \frac{1}{\sqrt{1+\lambda^2}}(e_1^h + \lambda\eta^v)$ form an orthonormal basis of $\tilde{\pi}$. So we may apply (5) setting $X_1 = e_0$, $X_2 = 0$, $Y_1 = \frac{1}{\sqrt{1+\lambda^2}}e_1$, $Y_2 = \frac{\lambda}{\sqrt{1+\lambda^2}}\eta$.

We get

$$\begin{aligned}
&\big\langle R(X_1,Y_1)Y_1,X_1\big\rangle = \frac{1}{1+\lambda^2}\big\langle R(e_0,e_1)e_1,e_0\big\rangle = \frac{1}{1+\lambda^2}K,\\
&\|R(X_1,Y_1)\xi\|^2 = \frac{1}{1+\lambda^2}\|R(e_0,e_1)\xi\|^2 = \frac{1}{1+\lambda^2}K^2,\\
&\|R(\xi,Y_2)X_1\|^2 = \frac{\lambda^2}{1+\lambda^2}\|R(\xi,\eta)e_0\|^2 = \frac{\lambda^2}{1+\lambda^2}K^2,\\
&\big\langle(\nabla_{X_1}R)(\xi,Y_2)Y_1,X_1\big\rangle = \frac{\lambda}{1+\lambda^2}\big\langle(\nabla_{e_0}R)(\xi,\eta)e_1,e_0\big\rangle = -(-1)^s\frac{\lambda}{1+\lambda^2}e_0(K),
\end{aligned}$$

where K is the Gaussian curvature of M. Applying directly (5) we obtain

$$\begin{aligned}
K_{T_1M}(\xi) \quad &= \frac{1}{1+\lambda^2}\left(K - \frac{3}{4}K^2 + \frac{\lambda^2K^2}{4} + (-1)^{s+1}\lambda e_0(K)\right)\\
&= \frac{1}{1+\lambda^2}\left(K(1-K) + \frac{(1+\lambda^2)K^2}{4} + (-1)^{s+1}\lambda e_0(K)\right)\\
&= \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} + (-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K).
\end{aligned}$$

■

Now we have the following.

Lemma 3.4 *Let ξ be a unit vector field on a 2-dimensional Riemannian manifold M. In terms of Lemma 3.1, the Gaussian curvature K_ξ of the hypersurface $\xi(M) \in T_1M$ is given by*

$$\begin{aligned}
K_\xi = \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} \quad &+(-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K) +\\
&\frac{1}{2}\mu e_1\left(\frac{1}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2,
\end{aligned}$$

where K is the Gaussian curvature of M.

Proof. In our case, one can easily reduce the formula (6) to the form

$$K_\xi = K_{T_1M}(\xi) + \det\Omega.$$

Applying Lemma 3.2, we see that

$$\begin{aligned}
\det\Omega = \quad &-\mu\frac{\lambda}{\sqrt{1+\lambda^2}}e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2 =\\
&-\frac{1}{2}\mu e_1\left(\frac{\lambda^2}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2 =\\
&\frac{1}{2}\mu e_1\left(\frac{1}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2.
\end{aligned}$$

Combining this result with Lemma 3.3, we get what was claimed.

■

As an application of Lemma 3.4 we prove the following theorems.

Theorem 3.2 *Let M^2 be a 2-dimensional Riemannian manifold of Gaussian curvature K. Suppose that ξ is a unit geodesic vector field on M^2. Then the submanifold $\xi(M^2) \subset T_1M^2$ has non-positive extrinsic curvature.*

Proof. By definition, the extrinsic curvature of a submanifold is the difference between the sectional curvature of the submanifold and the sectional curvature of ambient space along the planes, tangent to the submanifold. In our case , this is $\det \Omega$. If ξ is a geodesic vector field, then we may choose $e_0 = \xi$ and then $\mu = k = 0$. Therefore, for the extrinsic curvature we get

$$-\left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2 \le 0\,.$$

■

Theorem 3.3 *Let M^2 be a space of constant Gaussian curvature K. Suppose that ξ is a unit geodesic vector field on M^2. Then $\xi(M^2)$ has constant Gaussian curvature in one of the following cases:*

(a) $K = -c^2 < 0$ and ξ is a normal vector field for the family of horocycles on the hyperbolic 2-plane L^2. In this case $K_\xi = -c^2$ and therefore $\xi(L^2)$ is locally isometric to L^2;

(b) $K = 0$ and ξ is a parallel vector field on M^2. In this case $K_\xi = 0$ and $\xi(M^2)$ is also locally isometric to M^2;

(c) $K = 1$ and ξ is any (local) geodesic vector field on the standard sphere S^2. In this case $K_\xi = 0$.

Proof.

Since ξ is geodesic, we may set $e_0 = \xi,\ e_1 = \eta,\ s = 1$. Taking into account (7) and (8), we see that $\lambda = -\kappa = -\sigma$. Lemma 3.1 (b) gives $-K = -e_0(\sigma)+\sigma^2$. So the result of Lemma 3.4 takes the form

$$\begin{aligned} K_\xi = \quad & \frac{K^2}{4} + \frac{K(1-K)}{1+\sigma^2} - \left(\frac{K}{2} - \frac{e_0(\sigma)}{1+\sigma^2}\right)^2 = \\ & \frac{K^2}{4} + \frac{K(1-K)}{1+\sigma^2} - \left(\frac{K}{2} - \frac{K+\sigma^2}{1+\sigma^2}\right)^2 = \\ & \frac{K(1-K)}{1+\sigma^2} + \frac{K(K+\sigma^2)}{1+\sigma^2} - \left(\frac{K+\sigma^2}{1+\sigma^2}\right)^2 = \\ & K - \left(\frac{K+\sigma^2}{1+\sigma^2}\right)^2 . \end{aligned}$$

Suppose that K_ξ is constant. Then the following cases should be considered:

(a) $\sigma = const \neq 0$. This means that the orthogonal trajectories of the field ξ consist of curves of constant curvature. With respect to this natural coordinate system, the metric of M^2 takes the form $ds^2 = du^2 + f^2\,dv^2$. Set $\sigma = -c$. Then the function f should satisfy the equation

$$\frac{f_u}{f} = c$$

the general solution of which is $f(u,v) = A(v)e^{cu}$. After v-parameter change we obtain metric of the form

$$ds^2 = du^2 + e^{2cu}\,dv^2.$$

So, the manifold M^2 is locally isometric to the hyperbolic 2-plane L^2 of curvature $-c^2$ and the field ξ is a geodesic field of (internal or external) normals to the family of horocycles.

(b) $\sigma = 0$. Then evidently ξ is a parallel vector field and therefore the manifold M^2 is locally Euclidean which implies K_ξ=0.

(c) σ is not constant. Then K_ξ is constant if $K = 1$ only. So, M^2 is contained in a standard sphere S^2 and the curvature of $\xi(S^2)$ does not depend on σ. Thus, the field ξ is any (local) geodesic vector field. Evidently, $K_\xi = 0$ for this case.

■

The case (a) of the Theorem 3.3 has an interesting generalization of the following kind.

Theorem 3.4 *Let L^2 be a hyperbolic 2-plane of curvature $-c^2$. Then T_1L^2 admits a hyperfoliation with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$. The leaves are generated by unit vector fields making a constant angle with a pencil of parallel geodesics on L^2.*

Proof. Consider L^2 with metric $ds^2 = du^2 + e^{2cu}\,dv^2$ and a family of vector fields

$$\xi_\omega = \cos\omega X_1 + \sin\omega X_2 \quad (\omega = const),$$

where $X_1 = \{1, 0\}, \quad X_2 = \{0, e^{-cu}\}$ are the unit vector fields.

Since $\nabla_{X_1}\xi_\omega = 0$, we may set $e_0 = X_1, \quad e_1 = X_2$ end therefore we have $\sigma = -c, \quad \lambda = c$. Then, setting $K = -c^2$ and $\lambda = c$ in Lemma 3.4, we get

$$K_\xi = -c^2.$$

The extrinsic curvature of $\xi(L^2)$ is also constant since

$$\det\Omega = -\frac{1}{4}c^2.$$

Now fix a point P_∞ at infinity boundary of L^2 and draw a pencil of parallel geodesics from P_∞ through each point of L^2. Define a family of submanifolds $\xi_\omega(L^2)$ for this pencil. Evidently, through each point $(p,\zeta) \in T_1L^2$ there passes only one submanifold of this family. Thus, a family of submanifods ξ_ω form a hyperfoliation on T_1L^2 of constant intrinsic curvature $-c^2$ and constant extrinsic curvature $-\frac{c^2}{4}$.

Geometrically, $\xi_\omega(L^2)$ is a family of coordinate hypersurfaces $\omega = const$ in T_1L^2. Indeed, let (u, v, ω) form a natural local coordinate system on T_1L^2. Then the metric of T_1L^2 has the form

$$ds^2 = du^2 + 2e^{2u}dv^2 + 2\,dvd\,\omega + d\,\omega^2.$$

With respect to these coordinates, the coordinate hypersurface $\omega = const$ is nothing else but $\xi_\omega(L^2)$ and the induced metric is

$$ds^2 = du^2 + 2e^{2cu}dv^2.$$

Evidently, its Gaussian curvature is constant and equal to $-c^2$.

■

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